Constructing Quasi-Monte Carlo Points With and Without Sensitivity Analysis

Pierre L'Ecuyer
Gregory De Salaberry Seljak
Université de Montréal, Montréal, Canada

Good point sets for quasi-Monte Carlo (QMC) integration are usually constructed by selecting parameters to minimize a figure of merit (FOM) that measures the discrepancy between the empirical distribution of the points and the uniform distribution [1, 2, 6]. These FOMs often give weights to the different subsets of coordinates, to account for their relative importance, as done by the software in [7, 8], for example. The weights should reflect the variance contributions (or sensitivity indices) of these subsets, which are typically unknown and costly to estimate.

A much simpler alternative is to bypass these FOMs and simply draw the parameters of the QMC rule at random from some distribution. It turns out that for the popular QMC constructions, the probability of drawing bad parameters (that give a large RQMC variance) is pretty small.

With randomized QMC (RQMC), we randomize the points r times independently to compute r independent replicates of the unbiased RQMC estimator, and we usually take the empirical mean and variance of these r replicates to estimate the true mean (the integral) and perhaps compute a confidence interval [9]. When the QMC parameters are selected at random, independently for the r replications, it may be better to replace the empirical mean by a more robust estimator such as the median or something more refined, so that the outliers that may come from the rare unlucky parameter choices have little impact on the final estimator. This idea was proposed and studied recently in [3, 4, 5, 10, 11].

In this talk, we review these recent studies and we report on experiments that compare the mean square error (MSE) of various estimators (the mean, the median, and others) in RQMC settings. We also look at how to compute confidence intervals for the mean in these settings.

References:

[1] J. Dick, P. Kritzer, and F. Pillichshammer. *Lattice Rules: Numerical Integration, Approximation, and Discrepancy.* Springer, 2022.

[2] J. Dick and F. Pillichshammer. *Digital Nets and Sequences: Discrepancy Theory and Quasi-Monte Carlo Integration.* Cambridge University Press, Cambridge, U.K., 2010.

[3] E. Gobet, M. Lerasle, and D. Métivier. Accelerated convergence of error quantiles using robust randomized quasi Monte Carlo methods. Preprint, `https://hal.science/hal-03631879`, 2024.

[4] T. Goda and P. L'Ecuyer. Construction-free median quasi-Monte Carlo rules for function spaces with unspecified smoothness and general weights. *SIAM Journal on Scientific Computing*, 44(4):A2765–A2788, 2022.

[5] T. Goda, K. Suzuki, and M. Matsumoto. A universal median quasi-Monte Carlo integration. *SIAM Journal on Numerical Analysis*, 62(1):533–566, 2024.

[6] P. L'Ecuyer, Y. Cherkanihassani, and M. El Amine Derkaoui. Pre-scrambled digital nets for randomized quasi-Monte Carlo. In *Proceedings of the 2024 Winter Simulation Conference*, pages 443–454. IEEE Press, 2024.

[7] P. L'Ecuyer, P. Marion, M. Godin, and F. Puchhammer. A tool for custom construction of QMC and RQMC point sets. In A. Keller, editor, *Monte Carlo and Quasi-Monte Carlo Methods: MCQMC 2020*, pages 51–70, Berlin, 2022. Springer.

[8] P. L'Ecuyer and D. Munger. Algorithm 958: Lattice builder: A general software tool for constructing rank-1 lattice rules. *ACM Transactions on Mathematical Software*, 42(2):Article 15, 2016.

[9] P. L'Ecuyer, M. Nakayama, A. B. Owen, and B. Tuffin. Confidence intervals for randomized

quasi-Monte Carlo estimators. In *Proceedings of the 2023 Winter Simulation Conference*, pages 445–456. IEEE Press, 2023.

[10] Z. Pan and A. B. Owen. Super-polynomial accuracy of one dimensional randomized nets using the median of means. *Mathematics of Computation*, 92(340):805–837, 2023.

[11] Z. Pan and A. B. Owen. Super-polynomial accuracy of multidimensional randomized nets using the median of means. *Mathematics of Computation*, 93(349):2265–2289, 2024.

[Pierre L'Ecuyer; Université de Montréal, C.P. 6128, Succ. Centre-Ville, Montreal, Canada]
[`lecuyer@iro.umontreal.ca` – `https://www-labs.iro.umontreal.ca/~lecuyer/`]