Sample Average Approximation for Portfolio Optimization under CVaR constraint in an (re)insurance context

Jérôme, Lelong
Univ. Grenoble Alpes, CNRS, Grenoble INP, LJK, 38000 Grenoble, France.

Véronique Maume-Deschamps
William Thevenot
Universite Claude Bernard Lyon 1, CNRS, Ecole Centrale de Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France.

(Re)insurers are constantly looking for opportunities to develop their business, increase their incomes and improve their profitability. However, in every line of business, portfolio growth often leads to increased incomes and increased risk accumulation. The company's aim is to maximize profitability by achieving an optimal risk/reward ratio between exposure to losses and expected profits. Although, the assessment of individual risks is important, getting the right mix of risks is just as crucial.

In addition, on the European market, an insurer must meet the requirements of Solvency II regulations, in particular it must have an amount of own funds at least equal to the Solvency Capital Requirement (SCR). The SCR is the capital required to ensure that the (re)insurance company will be able to meet its obligations over the next 12 months with a probability greater than 99.5%. Formally, it is modeled with the Value-At-Risk (VaR) at the level $\alpha = 0.995$.

Other risk measures can be used to model the overall risk of a (re)insurance company, the most widely used alternative being the Conditional Value-At-Risk ($CVaR$), also called Tail Value-At-Risk (TVaR) or expected shortfall (ES) for continuous distributions. CVaR is usually preferred to VaR because it has better properties such as sub-additivity and its coherent in the sense of Artzner et al.[1]. It is in the company's interest to reduce risk through diversification, in order to achieve the best risk/return ratio.

The classic approach to portfolio optimization was introduced by Markowitz in 1952 [3]. It consists in the maximization of the expectation under the constraint of maximum variance or, equivalently, minimizing the variance of the portfolio, for a fixed return, this problem is called the mean-variance optimization. Its equivalent for the conditional value-at-risk (CVaR) is the mean-CVaR optimization.

We model the (re)insurance asset market with business lines represented by the random vector $\mathbf{X}$ of asset returns, taking values in a subset $\mathcal{R}_{\mathbf{X}}$ of $\mathbb{R}^d$. We assume that $\mathbb{E}(|\mathbf{X}|) < +\infty$. A (re)insurance portfolio is defined by a vector $\boldsymbol{\gamma} \in \mathbb{R}^d$ representing the quantity held in each business line by the (re)insurer.

Let us fix some notations, with $\alpha \in]0, 1[$:

$$V_\alpha(\boldsymbol{\gamma}) = VaR_\alpha(-\boldsymbol{\gamma}^T\mathbf{X}) = \min\left\{M \in \mathbb{R} : \mathbb{P}\left(-\boldsymbol{\gamma}^T\mathbf{X} \leq M\right) \geq \alpha\right\},$$
$$C_\alpha(\boldsymbol{\gamma}) = CVaR_\alpha(-\boldsymbol{\gamma}^T\mathbf{X}) = \mathbb{E}\left(-\boldsymbol{\gamma}^T\mathbf{X}\middle| -\boldsymbol{\gamma}^T\mathbf{X} \geq V_\alpha(\boldsymbol{\gamma})\right).$$

Our original goal is to solve the following equation with a fixed $\alpha \in]0, 1[$ and constraints on the weights and a capital requirement limit $K > 0$. It is quite common that L_0 depends on $C_\alpha(\boldsymbol{\gamma})$.

$$\begin{aligned} v^* := \inf_{\boldsymbol{\gamma}\in\mathbb{R}^d_+} \quad & \mathbb{E}(L_0(\boldsymbol{\gamma}, C_\alpha(\boldsymbol{\gamma}), \mathbf{X})) \\ \text{s.t.} \quad & \gamma_i^{low} \leq \gamma_i \leq \gamma_i^{up} \qquad \forall i \in \{1, .., d\} \\ \text{s.t.} \quad & C_\alpha(\boldsymbol{\gamma}) \leq K. \end{aligned} \tag{1}$$

A new approach was introduced by R.T. Rockafellar and S. Uryasev in 2000 [4] and was later extended by Krokhmal P., Jonas Palmquist J., Uryasev S. (2002) [2] who proposed an embedding technique to reformulate the CVaR.

We aim to maximize a return function or minimize a loss function of a portfolio under CVaR constraints, because this approach is well adapted to the needs of (re)insurance companies. In [2], it is solved using linear programming, but this resolution can be very time-consuming. In this work, we prefer to use *Sample Average Approximation* (SAA), see Rubinstein and Shapiro [5].

For this formulation with explicit constraints, no convergence or convergence speed results with the SAA method has been published as far as we know, the closest result to our work is [8]. In this last one, the function to be minimized does not depend on the data sample.

Under convexity, continuity, integrability assumptions, we prove a.s. the convergence and find a rate of convergence for the SAA version in the case where the function to be minimized depends on the data sample as do the constraint. Moreover, if the CVaR appears in the function to be minimized, we show that for the optimization, under monotonic assumption, it can be replaced by the auxiliary function introduced in [2] and [4]. We also propose a sufficient condition to obtain the uniqueness of the solution. These results give (re)insurers a practical solution to portfolio optimization under market regulatory constraints, i.e. a certain level of risk.

References:

[1] Philippe Artzner et al. "Coherent measures of risk". *Mathematical finance* 9.3 (1999), pp. 203–228.

[2] Pavlo Krokhmal, Jonas Palmquist, and Stanislav Uryasev. "Portfolio optimization with conditional value-at-risk objective and constraints". *Journal of risk* 4 (2002), pp. 43–68.

[3] Harry Markowitz. "Modern portfolio theory". *Journal of Finance* 7.11 (1952), pp. 77–91.

[4] R Tyrrell Rockafellar, Stanislav Uryasev, et al. "Optimization of conditional value-at-risk". *Journal of risk* 2 (2000), pp. 21–42.

[5] Reuven Y Rubinstein and Alexander Shapiro. "Discrete event systems: sensitivity analysis and stochastic optimization by the score function method." Vol. 13. Wiley, 1993.

[6] Alexander Shapiro. "Monte Carlo sampling approach to stochastic programming". *ESAIM*: proceedings. Vol. 13. EDP Sciences. 2003, pp. 65–73.

[7] Alexander Shapiro, Darinka Dentcheva, and Andrzej Ruszczynski. "Lectures on Stochastic Programming: Modeling and Theory" (2009).

[8] Wei Wang and Shabbir Ahmed. "Sample average approximation of expected value constrained stochastic programs". *Operations Research Letters* 36.5 (2008), pp. 515–519.

[William Thevenot; Universite Claude Bernard Lyon 1, CNRS, Ecole Centrale de Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France. and Risk Knowledge team at SCOR SE, Paris, France]
[`thevenot@math.univ-lyon1.fr` –]

[Véronique Maume-Deschamps; Universite Claude Bernard Lyon 1, CNRS, Ecole Centrale de Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France.]
[`veronique.maume-deschamps@univ-lyon1.fr` –]

[Jérôme Lelong; Univ. Grenoble Alpes, CNRS, Grenoble INP, LJK, 38000 Grenoble, France.]
[`jerome.lelong@univ-grenoble-alpes.fr` –]